Explicit Eigenvalue Approach to the Efficient Determination of the Hybrid Spectrum of Ferrite-Loaded Circular Waveguide

Luca Pierantoni, Alessandro Camicia, and Tullio Rozzi

Dipartimento di Elettronica ed Automatica, Università degli Studi di Ancona, 60131 Ancona, Italy

Abstract —**Finding many higher modes in ferrite loaded waveguide becomes mandatory when dealing with discontinuities in accurate component design. It is well known, however, that this is a very difficult task in practice. In this contribution we report a novel approach to this problem based on the explicit eigenvalue formulation of the vector telegrapher's equations applied to the anisotropic uniform waveguide and discretized by using the actual modes taken at cut-off as an expanding set. Theoretical results are compared with existing ones for the case of longitudinally magnetized fully and partially loaded waveguide with very good agreement. The resulting algorithm is compact and it requires extremely low computational effort for evaluating as many modal characteristics and vector fields as required.**

I. INTRODUCTION

Cylindrical waveguides containing longitudinally magnetized ferrite are widely used in microwave devices such us circulators, isolators, phase shifters and control components, [1], [2]. The anisotropic material in such devices exhibits non-reciprocal behavior in presence of an external bias magnetic field, [3]. Knowledge of the propagation characteristics such as cutoff planes and phase constants of many modes is of considerable importance in the accurate design of actual components. An extensive analysis of cut-off phenomena in a longitudinally-magnetized ferrite-filled circular waveguide is reported in [4]-[6]. The analysis of the propagation in fully and partially filled waveguide is reported in [3], where the vector solutions are derived from a pair of coupled wave equations for the longitudinal electric and magnetic field components. The difficulty in [3] arises in actually solving numerically an implicit trascendental characteristic equation with spurious solutions and crossing dispersion curves that are hard to follow. In [7] cut-off and phase constants of partially filled axially magnetized ferrite-loaded guides are calculated using the numerical Finite Element Method (FEM); the disadvantage here occurs in the extremely heavy computational effort. In both cases, the derivation is practically limited to the first few higher order modes.

The goal of the present approach is an efficient and compact algorithm for the accurate spurious-free determination of propagation characteristics and behavior of very many hybrid modes. The method leads to an explicit and analytical formulation of the eigenvalue equation, thus avoiding the limits of [3] and [7]. An analogous approach has been introduced in [8] for the analysis planar structures of complex shape. The essential step of the method is the derivation of generalized vector telegrapher's equation governing the propagation of the transverse field in waveguides loaded by anisotropic media. In the past, generalized vector telegrapher's equations have only been derived only for isotropic media [9].

The telegrapher's equation are then discretized by expanding the transverse field in terms of the actual modes taken at cutoff, which constitute a set of frequency-independent real vector fields implicitly satisfying all boundary and edge conditions of the problem. These are in fact the 3-component modes of a 2D-resonator coinciding with the guide cross-section. The above position leads to an explicit linear eigenvalue matrix equation that is easily solved with minor computational effort. Numerical results produced by the present method are then compared with data reported in literature, showing very good agreement.

II. THEORY

We report without derivation the field equations for the propagation of the transverse field along the guide, in absence of sources:

$$-\frac{\partial}{\partial z}\mathbf{H}_t = -\nabla_t \frac{1}{jw m_0 m_{zz}} \nabla_t \cdot \left(\hat{\mathbf{z}}\times\mathbf{E}_t\right) - \nabla_t\left[\frac{\underline{m}_{zt}}{m_{zz}}\mathbf{H}_t\right] + jwe_0e_r\left(\hat{\mathbf{z}}\times\mathbf{E}_t\right) \quad (1a)$$

$$-\frac{\partial}{\partial z}\mathbf{E}_t = jw m_0\left(\underline{m}_{tt}\mathbf{H}_t\times\hat{\mathbf{z}}\right) - \nabla_t\left[\frac{1}{jwe_0e_r}\nabla_t\cdot\left(\mathbf{H}_t\times\hat{\mathbf{z}}\right)\right] + \left[\frac{\underline{m}_{tz}}{m_{zz}}\nabla_t\cdot\left(\hat{\mathbf{z}}\times\mathbf{E}_t\right)\right]\times\hat{\mathbf{z}} - jw\frac{m_0}{m_{zz}}\left(\underline{m}_{tz}\underline{m}_{zt}\mathbf{H}_t\times\hat{\mathbf{z}}\right) \quad (1b)$$

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where: $\mathbf{E_t}(x,y,z)=\mathbf{E_t}(x,y)e^{-\gamma z}$, $\mathbf{H_t}(x,y,z)=\mathbf{H_t}(x,y)e^{-\gamma z}$ are the trasverse components (respect to the z-direction) of the hybrid field. We consider the permeability tensor $\underline{\mu}$ in its general form, with 3x3 components:

$$\underline{m} = m_0\underline{m}_{\mathbf{r}} \qquad \underline{m}_{\mathbf{r}} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix}$$

In eqs.(1) $\underline{\mu}_{zt}$ is the row vector (μ_{zx}, μ_{zy}), $\underline{\mu}_{tz}$ is the column vector (μ_{xz}, μ_{yz}) and $\underline{\mu}_{tt}$ is the transverse-to-transverse block of $\underline{\mu}_r$. Dually, generalized telegrapher's equation for the tensor $\underline{\varepsilon}$ can be derived.

By considering the longitudinal magnetization as in [3] the tensor $\underline{\mu}_r$ reduces to:

$$\underline{m}_{\mathbf{r}} = \begin{bmatrix} m & jk_p & 0 \\ -jk_p & m & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

being: $m = 1 + \frac{w_0 w_m}{w_0^2 - w^2}$, $\qquad k_p = \frac{w w_m}{w_0^2 - w^2}$.

Setting in (1) $\underline{\mu}_{\mathbf{tz}}=\mathbf{0}$, $\underline{\mu}_{zt}=\mathbf{0}$ we derive:

$$-\frac{\partial}{\partial z}\mathbf{H_t} = jwe_0e_r\left(\hat{\mathbf{z}}\times\mathbf{E_t}\right) - \nabla_{\mathbf{t}}\frac{1}{jwm_0m_{zz}}\nabla_{\mathbf{t}}\cdot\left(\hat{\mathbf{z}}\times\mathbf{E_t}\right) \qquad (2a)$$

$$-\frac{\partial}{\partial z}\mathbf{E_t} = jwm_0\left(\underline{m}_{tt}\mathbf{H_t}\times\hat{\mathbf{z}}\right) - \nabla_{\mathbf{t}}\frac{1}{jwe_0e_r}\nabla_{\mathbf{t}}\cdot\left(\mathbf{H_t}\times\hat{\mathbf{z}}\right) \qquad (2b)$$

with
$$\underline{m}_{tt} = \begin{bmatrix} m & jk_p \\ -jk_p & m \end{bmatrix} \qquad (3)$$

In the case of partially or totally ferrite filled waveguide propagation is hybrid in general. The resulting modes are called: i) EH-modes if E_z is dominant, ii) HE-modes if H_z is dominant, [3]. We now select the actual modes taken at cut-off as the expanding set for the transverse field by setting $\gamma=0$ in eq.(1). Consequently, the hybrid modes propagating in the normal z-direction reduce to pure real TE/TM modes. In particular, HE modes reduce to TE modes and EH modes reduce to TM modes. In fact, at cut-off:

$$\mathbf{e_t}(x,y) = \mathbf{e_t}^{TE}(x,y) \qquad e_z^{TE}(x,y) = 0 \qquad (4a)$$

$$\mathbf{h}_t(x,y) = \mathbf{h_t}^{TM}(x,y) \qquad h_z^{TM}(x,y) = 0 \qquad (4b)$$

Now we expand the transverse fields as:

$$\mathbf{E_t}(w) = \sum_{k\in TM} V_k^e(w)(\mathbf{h}_{\mathbf{t}_k}\times\hat{\mathbf{z}}) + \sum_{k\in TE} V_k^h(w)\mathbf{e}_{\mathbf{t}_k} \qquad (5a)$$

$$\mathbf{H_t}(w) = \sum_{k\in TM} I_k^e(w)\mathbf{h}_{\mathbf{t}_k} + \sum_{k\in TE} I_k^h(w)(\hat{\mathbf{z}}\times\mathbf{e}_{\mathbf{t}_k}) \qquad (5b)$$

where the *k*-index spans the TE (*h*) modes and the TM (*e*) modes. $\boldsymbol{e}_{t_k}$, $\boldsymbol{h}_{t_k}$ are real and frequency-independent whereas the expansion coefficients V_k^e, V_k^h, I_k^e, I_k^h contain the frequency dependence. It can be also be seen from (2) that the TE/TM components at cutoff are solutions of the following transverse equations:

$$\nabla_{\mathbf{t}}\frac{1}{jwm_0m_{zz}}\nabla_{\mathbf{t}}\cdot(\hat{\mathbf{z}}\times\mathbf{e}_{\mathbf{t}_k}^{TE}) = j\frac{w_{c_k}^2}{w}e_0e_r(\hat{\mathbf{z}}\times\mathbf{e}_{\mathbf{t}_k}^{TE}) \qquad (6a)$$

$$\nabla_{\mathbf{t}}\frac{1}{jwe_0e_r}\nabla_{\mathbf{t}}\cdot(\mathbf{h}_{\mathbf{t}_k}^{TM}\times\hat{\mathbf{z}}) = j\frac{w_{c_k}^2}{w}m_0(\underline{m}_{\mathbf{tt}}\mathbf{h}_{\mathbf{t}_k}^{TM}\times\hat{\mathbf{z}}) \qquad (6b)$$

$$\nabla_{\mathbf{t}}\cdot\mathbf{e}_t^{TE} = 0 \qquad (7)$$

Now by substituting eqs.(5) in (2) and exploiting the above properties (6) and (7), we derive a particularly simple, discretized matrix version of (2).

$$g\,\underline{\mathbf{A}}\begin{bmatrix}\mathbf{I}^h \\ \mathbf{I}^e\end{bmatrix} = jwe_0\left\{\underline{\mathbf{C}} - \frac{1}{w^2}\underline{\mathbf{C}}\begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_e\end{bmatrix} + \frac{1}{w^2m_0e_0}\underline{\mathbf{D}}\right\}\begin{bmatrix}\mathbf{V}^h \\ \mathbf{V}^e\end{bmatrix} \qquad (9a)$$

$$g\,\underline{\mathbf{A}}\begin{bmatrix}\mathbf{V}^h \\ \mathbf{V}^e\end{bmatrix} = jwm_0\left\{\underline{\mathbf{F}} - \frac{1}{w^2}\underline{\mathbf{G}}\begin{bmatrix}\Omega_h & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{I}^h \\ \mathbf{I}^e\end{bmatrix} \qquad (9b)$$

$$\Omega_e = diag(w_{c_{e1}}^2,...,w_{c_{en}}^2) \qquad \Omega_h = diag(w_{c_{h1}}^2,...,w_{c_{hn}}^2)$$

where $\omega_{c_{ek}}$, $\omega_{c_{hk}}$ is the k-cutoff frequency for the k- TM/TE component, respectively. $\underline{\mathbf{A}}$, $\underline{\mathbf{C}}$, $\underline{\mathbf{D}}$, $\underline{\mathbf{F}}$ and $\underline{\mathbf{G}}$ are square-matrices (n x n) obtained by forming the scalar products between ($\boldsymbol{h}_{tk}^*$ x $\underline{z}$), $\boldsymbol{e}_{tk}$, (k=1,…,n) and the discretized equation version of (2a) and (2b) x $\underline{z}$, respectively. Surface integration is performed over the cross section S. The above real matrices represent the overlapping of the expanding cut-off modes. For example:

$$\underline{\mathbf{A}} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix} \text{ being } \quad \mathbf{A}_{11}^{m,p} = \iint_S \mathbf{h}_{tm}^* \cdot \mathbf{h}_{tp}\,dS$$

$$\mathbf{A}_{12}^{m,p} = \iint_S \left(\mathbf{h}_{tm}^*\times\hat{z}\right)\cdot\mathbf{e}_{tp}\,dS \qquad \mathbf{A}_{22}^{m,p} = \iint_S \mathbf{e}_{tm}^*\cdot\mathbf{e}_{tp}\,dS$$

$\mathbf{A}_{21}$ is the Hermitian conjugate of $\mathbf{A}_{12}$. Each of the two systems (9a) and (9b) is constituted by 2n-linear equations for the 4n unknown expanding coefficients V_k^h, V_k^e, I_k^h, I_k^e, (k=1..n); therefore (9a) and (9b) can be combined in order to form a single system characterized by 2n equations for V_k^h, V_k^e, (or I_k^h, I_k^e); its determinant is a polynomial equation for the square of the complex propagation constant, γ^2. Now, if we pre-multiply (9a) by $\underline{\mathbf{A}}^{-1}$ (the inverse of $\underline{\mathbf{A}}$, assumed non-singular) divide both members by γ and substitute $\underline{\mathbf{I}}$ from the above equation in (9b), we obtain:

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$$\frac{g^2}{k_0^2}\underline{\mathbf{A}}\begin{bmatrix}\mathbf{V}^h\\ \mathbf{V}^e\end{bmatrix} = -\underline{\mathbf{Q}}\begin{bmatrix}\mathbf{V}^h\\ \mathbf{V}^e\end{bmatrix} \qquad (10)$$

where:

$$\underline{\mathbf{Q}} = \left\{\underline{\mathbf{F}} - \frac{1}{w^2}\underline{\mathbf{G}}\begin{bmatrix}\Omega_h & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\}\underline{\mathbf{A}}^{-1}\cdot$$

$$\cdot\left\{\underline{\mathbf{C}} - \frac{1}{w^2}\underline{\mathbf{C}}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \Omega_e\end{bmatrix} + \frac{1}{w^2 m_0 e_0}\underline{\mathbf{D}}\right\} \qquad (11)$$

In the above eq.(10) we recognize a generalized eigenvalue problem for γ^2, whose derivation was our main concern. Once the frequency is fixed, each eigenvalue $\gamma^2(\omega)$ also yields the corresponding eigenvector $\underline{\mathbf{V}}$, which, in turn, gives the transverse modal electric field $\mathbf{E_t}$ in (5a). The other eigenvector, $\underline{\mathbf{I}}$ producing $\mathbf{H_t}$ in (5b), can be easily derived from (9a). The longitudinal components E_z and H_z are evaluated in terms of the transverse field, as in [8].

III. Results

The method described above is now applied to the problem of finding cutoff planes and phase constants for various TE and TM modes of cylindrical waveguides containing longitudinally magnetized ferrite. Fig.1 shows the analyzed ferrite loaded waveguide with electric walls and radius a. As discussed in the previous section, the first step of the method is an initial calculation and storage of cutoff frequencies and fields of the TE/TM modes. This is done by setting a priori γ=0 in the curl equations and solving two decoupled scalar wave equations. In fig. 2, we report the cutoff planes for the first three modes; we have $k_0 a\varepsilon_f^{1/2}$ versus k_p/μ, being ε_f the dielectric constant of the ferrite and k_p, μ elements of the the tensor $\underline{\boldsymbol{\mu}}$ defined in (1). We observe that HE_{mn}, (TE_{mn}) modes are independent of the magnetization, while EH_{mn} (TM_{mn}) modes have cutoff numbers increasing with k_p. The results obtained by the present method are compared with those reported in [3], [7], with very good agreement. Now we insert the above calculated cutoff modes in (5) and solve eq.(10). We then use its solutions in the transverse field expansion (5) in order to form the "modes" of the present method. The simplest choice is to insert a single TE or TM mode (k=1) in order to reconstruct the dispersion curves reported in the literature and compare results. In Fig.3 we calculate the phase constant for the dominant HE_{11} mode, (β/k_0 versus k_p/μ) The results are compared to the ones in [7], with very good agreement. An important property of the waveguide being analyzed is that while the phase constant of its dominant HE_{11} mode splits in two curves (denoted as HE_{11}, HE_{11}), the corresponding cutoff number does not.

Fig.4 shows the phase constants for the second and third modes with k_0a=0.75, where k_0 is the free-space wave number. Fig.5 shows the first three modes with k_0a=1, respectively. Now we analyze the partially-filled waveguide. In Fig.6 we report the propagation curves (β/k_0 versus b/a) as a function of the parameter k_p, (μ=1), being a and b the radii of the waveguide and of the loading ferrite, respectively. It is noteworthy to note that all curves together are calculated in the computing time of some CPU-seconds on a normal 500 MHz PC. Neither spurious solutions are present nor difficulty is found in tracking modal dispersion curves.

IV. Conclusions

The authors present a novel approach to the full-wave analysis of the propagation in ferrite-loaded waveguide. This is based on the explicit eigenvalue formulation of the generalized vector telegrapher's equations applied to the anisotropic uniform waveguide, and discretized by using the modes taken at cut-off as an expanding set. The resulting algorithm is theoretically compact, efficient and able to find and trace very many dispersion curves and fields with extremely low computational CPU-time and no spurious solutions.

Theoretical results are compared with data in literature, for the first modes, with very good agreement.

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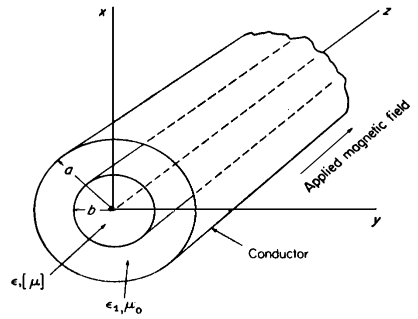

Fig.1.Cylindrical waveguide of radius a containing a co-axial ferrite cylinder of radius b, longitudinally magnetized.

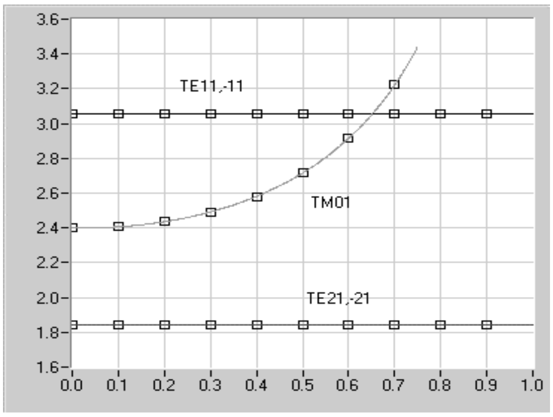

Fig. 2 Cutoff planes for the first three TE/TM modes ($k0a\varepsilon_f^{1/2}$ versus k_p/μ) calculated by the present method (continuos lines) and compared with those from [6], (squares).

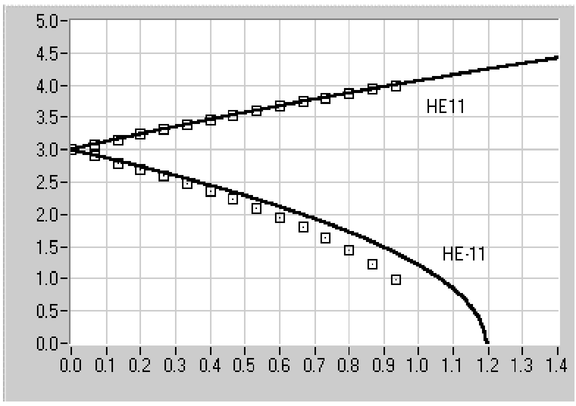

Fig. 3 Phase constant for the dominant HE_{11} mode, (β/k_0 -k_p/μ) compared with data in [6]. Present method: continuous line. Data from [6]: squares.

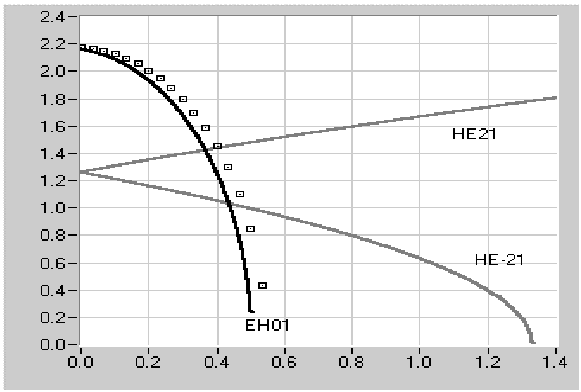

Fig.4. Phase constants for the second and third modes with k_0a=0.75. Present method: continuous lines. Data from [6]: squares.

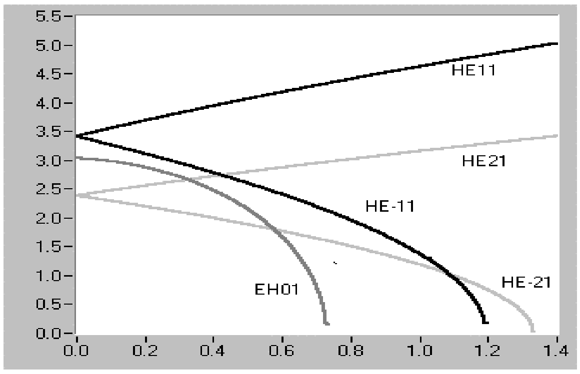

Fig.5. Phase constants for the first three modes with k_0a=1. Present method: continuous lines.

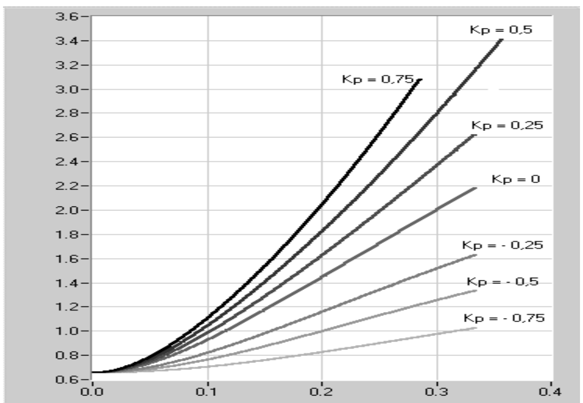

Fig.6 Partially filled waveguide. Propagation curves (β/k_0 versus b/a) as a function of k_p, (μ=1).

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